Casimir energy of a massive field in a genus-1 surface

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Abstract

We review the definition of the Casimir energy steming naturally from the concept of functional determinant through the zeta function prescription. This is done by considering the theory at finite temperature and by defining then the Casimir energy as its energy in the limit $T \to 0$. The ambiguity in the coefficient $C_{d/2}$ is understood to be a result of the necessary renormalization of the free energy of the system. Then, as an exact, explicit example never calculated before, the Casimir energy for a massive scalar field living in a general (1+2)-dimensional toroidal spacetime (i.e., a general surface of genus one) with flat spatial geometry —parametrized by the corresponding Teichmüller parameters— and its precise dependence on these parameters and on the mass of the field is obtained under the form of an analytic function.

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1 Introduction

The Casimir effect [1] is a beautiful and simple manifestation of the influence that boundaries or non-trivial spacetime topologies have on quantum field theories (see for example [2, 3, 4, 5]). The modern approach for the calculation of the Casimir energy is the zeta function regularization scheme put forward in [6]. The idea is the following. In order to have a well-defined notion of energy, let us work in a d-dimensional ultrastatic spacetime [7]. Thus the metric in some coordinate system is of the form $g = -(dx^0)^2 + g_S$ with the spatial part g_S of the metric (see Eq. (2.1)). The differential operator D describing the field equation may be decomposed as $D = -\partial_0^2 + D_S$. Introducing $E_n^2 = \lambda_n$ as the eigenfrequencies of D_S, the zero-point energy is formally given by

$$E_{Casimir} = \frac{1}{2}\sum_n E_n. \tag{1.1}$$

It may be regularized by defining

$$E_{reg}(\epsilon) = \frac{1}{2}\mu^{2\epsilon}\zeta_S(-1/2+\epsilon), \tag{1.2}$$

with $\zeta_S(s)$ being the zeta function associated with the $(d-1)$-dimensional operator D_S. The scale μ with dimension (length)$^{-1}$ has to be introduced in order to keep the zeta function dimensionless for all s.

General zeta function theory [8] tells us, that $E_{reg}(\epsilon)$ is a meromorphic function with a pole at $\epsilon = 0$, its residue being $-(1/2)C_{d/2}(D_S)/(4\pi)^{d/2}$. The coefficient $C_{d/2}(D_S)$ is the Seeley-De Witt coefficient appearing in the asymptotic expansion for small t of the heat-kernel associated with D_S,

$$\begin{aligned} K(t) &= \sum_n e^{-\lambda_n t} \\ &\sim \left(\frac{1}{4\pi t}\right)^{(d-1)/2} \sum_{l=0,1/2,1,\ldots}^{\infty} C_l(D_S)t^l. \end{aligned} \tag{1.3}$$

The pole appearing in Eq. (1.2) has to be absorbed into the bare action which thus must contain a term proportional to $C_{d/2}(D_S)$. It is clear then, that the Casimir energy has an ambiguity proportional to $C_{d/2}(D_S)$ (we will come back to this point later).

Adopting the minimal subtraction scheme, one defines

$$\begin{aligned} E_{Casimir} &= \frac{1}{2}\lim_{\epsilon\to 0}\frac{1}{2}\mu^{2\epsilon}\left[\zeta_S(-1/2+\epsilon)+\zeta_S(-1/2-\epsilon)\right] \\ &= \frac{1}{2}\left[PP\ \zeta_S(-1/2) - \frac{C_{d/2}(D_S)}{(4\pi)^{d/2}}\ln\mu^2\right], \end{aligned} \tag{1.4}$$

where the symbol PP stands for taking the principal part. Based on this definition, during the last years the Casimir energy has been calculated for a variety of examples [4, 9, 10].

The definition (1.4) of the Casimir energy is completely equivalent to the one steming naturally from the definition of functional determinant by the zeta function prescription (for very recent considerations on this issue see [11]). This may be done by considering the theory at finite temperature and by defining the Casimir energy as its energy in the limit $T \to 0$, this idea going back to Gibbons [12] who considered the single quantum mechanical oscillator. In the more general context of quantum field theory under some external conditions like boundaries or gravitational fields this definition has been employed for example in [13, 3, 14, 15, 16]). The ambiguity in the coefficient $C_{d/2}$ may be understood to be a result of the necessary renormalization of the free energy of the system.

Having summarized the main arguments in favor of the definition of the Casimir energy as given by Eq. (1.4), in section 3 we will present the calculation of the Casimir energy for a massive scalar field in a general $(1+2)$-dimensional toroidal spacetime with flat spatial geometry. The general flat geometry will be parametrized by the corresponding two Teichmüller parameters and the complete dependence of the Casimir energy on these parameters and on the mass of the field will be obtained under the form of an analytic function, by using the extended Chowla-Selberg zeta function formula derived by one of us in Refs. [17, 18]. For the massless case we obtain complete agreement with previous results by Dowker [19].

2 Zeta function definition of the Casimir energy

Let us first briefly summarize the motivation for the definition (1.4) of the Casimir energy. For definiteness, let us consider the quantum field theory of a free scalar field in curved spacetime, the Dirac field may be treated analogously. As mentioned before, in order to have a well defined notion of energy we shall restrict our considerations to a d-dimensional ultrastatic spacetime $\mathcal{M}$, possibly with a boundary, and with the metric

$$ds^2 = -dx_0^2 + g_{ab}(\vec{x})dx^a dx^b, \tag{2.1}$$

where $\vec{x} = (x_1, ..., x_{d-1})$. The action of the field theory we consider is [7], [20],

$$S = -\frac{1}{2} \int_{\mathcal{M}} d^d x |g|^{\frac{1}{2}} \phi^\dagger(x) \left(\Box - \xi R - m^2 \right) \phi(x), \tag{2.2}$$

with $\Box$ being the Laplace-Beltrami operator of the ultrastatic spacetime. Variation of equation (2.2) subject to the constraints

$$\begin{aligned} \delta\phi(x') &= 0, \\ n^{\mu'} \nabla_{\mu'} \delta\phi(x') &= 0, \end{aligned}$$

where the prime refers to quantities defined on the boundary $\partial\mathcal{M}$, yields the equation of motion

$$\left(\Box - \xi R - m^2 \right) \phi(x) = 0, \tag{2.3}$$

which is the generalized Klein-Gordon equation. The following discussion will be quite general, so the boundary condition need not to be specified at this point. A unique boundary value problem is posed, for example, by assuming Dirichlet- or Robin-boundary conditions on the field.

In an ultrastatic spacetime the quantum field theory at finite temperature may be developed in complete analogy with the Minkowski-space theory and that is why we will skip the details of the calculation. In the Euclidean formulation of the finite temperature theory, the partition sum $\mathcal{Z}$ may be written under the form of a functional integral of the kind

$$\mathcal{Z}[\beta] = \int [d\varphi] \exp\left\{ -\frac{1}{2}(\varphi, D\varphi) \right\}, \tag{2.4}$$

where we have used the scalar product

$$(f,h) = \int_0^\tau d\tau \int_\Sigma d\Sigma \; |g|^{1/2} f^* h \tag{2.5}$$

for the two vectors f and h, and being Σ the spatial section of the manifold $\mathcal{M}$. Here the integration extends over all fields periodic in the imaginary time τ with periodicity $\beta = 1/T$ and fulfilling the boundary conditions at the spatial boundary. The operator D is given by

$$D = -\frac{\partial^2}{\partial \tau^2} - \Delta + \xi R + m^2. \tag{2.6}$$

Then, one formally obtains

$$\mathcal{Z}[\beta, \mu] = (\det \lambda^{-2} D)^{-\frac{1}{2}}, \tag{2.7}$$

the scale λ [21] being necessary in order to keep everything dimensionless. The functional determinant of the operator $\lambda^{-2} D$ needs, of course, regularization. We will use the zeta-function regularization scheme introduced by Dowker, Critchley [22] and Hawking [21]. In this scheme, the free energy is defined as

$$F[\beta] = -\frac{1}{2\beta} \left[\zeta_d(0,\beta) \ln \lambda^2 + \zeta_d'(0,\beta) \right]. \tag{2.8}$$

The function $\zeta_d(s,\beta)$ is the zeta-function associated with the operator D. Using the ansatz

$$u_{l,k} = \frac{1}{\beta} \exp\left(\frac{2\pi i l}{\beta} \right) g_k(\vec{x})$$

the eigenvalues are seen to have the form

$$\nu_{l,k}^{\pm} = \left(\frac{2\pi l}{\beta} \right)^2 + E_k^2, \quad l \in \mathbb{Z},$$

with the energy eigenvalues being defined through

$$(-\Delta + \xi R + m^2)\psi_k(\vec{x}) = E_k^2 \psi_k(\vec{x}). \tag{2.9}$$

Making use of a Mellin-transformation and a theta-function identity [23], the free energy may be written in the form

$$\begin{aligned} F[\beta] &= \frac{1}{2}PP\zeta_S\left(-\frac{1}{2}\right)+\frac{1}{2(4\pi)^{\frac{d}{2}}}C_{\frac{d}{2}}[\ln\lambda^2-1+2\ln 2] \\ &+ \frac{1}{\beta}\sum_j \ln\left(1-e^{-\beta E_j}\right). \end{aligned} \tag{2.10}$$

The energy of the system is then given by

$$E=\frac{\partial}{\partial\beta}\beta F[\beta]. \tag{2.11}$$

Finally, defining the Casimir-energy as the limit of (2.11) for $T\to 0$, we obtain

$$E_{Cas}[\partial\mathcal{M}]=\lim_{T\to 0}E=\frac{1}{2}PP\zeta_S\left(-\frac{1}{2}\right)+\frac{1}{2(4\pi)^{\frac{d}{2}}}C_{\frac{d}{2}}\ln\tilde{\lambda}^2, \tag{2.12}$$

with $\tilde{\lambda}=\frac{2\lambda}{\sqrt{e}}$. We thus arrive to the definition of Refs. [14, 6]. In the last reference a detailed discussion of the meaning of this definition and of the problem of renormalization has been carried out. From the above derivation of the definition of the Casimir energy it is completely clear that the ambiguity of the Casimir energy is simply a result of the (in general) necessary renormalization of the free energy. In the case when the coefficient $C_{d/2}$ vanishes, the definition (2.12) gives a well defined, finite value.

3 Casimir energy in a (1+2)-dimensional toroidal spacetime

Let us consider, as an example, a (1+2)-dimensional spacetime with the topology $\mathrm{R}\times T^2$ [24]. We will concentrate on the case when the geometry of the space $\Sigma\simeq T^2$ is locally flat. One can construct such geometry, as is usual, by taking $\Sigma=[0,1]\times[0,1]/\sim$, where the equivalence relation is defined by $(\xi_1,0)\sim(\xi_1,1)$ and $(0,\xi_2)\sim(1,\xi_2)$. A flat 2-geometry is endowed on Σ by giving it a metric

$$ds^2=h_{ab}d\xi^a d\xi^b, \tag{3.1}$$

where

$$h_{ab}=\frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}. \tag{3.2}$$

The (τ_1,τ_2) are the Teichmüller parameters, independent of the spatial coordinates (ξ_1,ξ_2), and $\tau=\tau_1+i\tau_2$, $\tau_2>0$ [25, 26].

The Laplace-Beltrami operator of this metric is given by

$$\Delta=-\frac{1}{\tau_2}(|\tau|^2\partial_1^2-2\tau_1\partial_1\partial_2+\partial_2^2), \tag{3.3}$$

being its eigenvalues

$$\lambda_{n_1,n_2} = \frac{4\pi^2}{\tau_2}(|\tau|^2 n_1^2 - 2\tau_1 n_1 n_2 + n_2^2). \tag{3.4}$$

In the massive case, $m \neq 0$ the spectrum runs over $n_1, n_2 \in \mathbb{Z}$. In the massless case the zero-mode of Δ, $n_1 = n_2 = 0$, has to be excluded.

An exact analysis of the Casimir energy for this spacetime is possible since it reduces to a case of the Chowla-Selberg zeta function (when $m = 0$) or to one of the extended formula that has been obtained recently (case $m \neq 0$).

In fact, in the case when $m \neq 0$, the relevant formula is a particular application of the following. Let us consider the double series

$$E(s; a, b, c; q) \equiv \sum_{m,n\in\mathbf{Z}}{}' (am^2 + bmn + cn^2 + q)^{-s}, \tag{3.5}$$

with $q \neq 0$ (in general), the parenthesis in (3.5) is the inhomogeneous quadratic form

$$Q(x, y) + q, \qquad Q(x, y) \equiv ax^2 + bxy + cy^2, \tag{3.6}$$

restricted to the integers. In the general theory that deals with the homogeneous case, one assumes that $a, c > 0$ and that the discriminant

$$\Delta = 4ac - b^2 > 0 \tag{3.7}$$

(see [27]). Here we will impose the additional condition that q be such that $Q(m, n) + q \neq 0$, for all $m, n \in \mathbf{Z}$. In the usual applications of the theory, those conditions are indeed satisfied. For the analytical continuation of (3.5), the following expression has been obtained in Refs. [17, 18]

$$E(s; a, b, c; q) = 2\zeta_{EH}(s, q/a)\, a^{-s} + \frac{2^{2s}\sqrt{\pi}\, a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\,\Gamma(s-1/2)\zeta_{EH}(s-1/2, 4aq/\Delta) \tag{3.8}$$

$$+\frac{2^{s+5/2}\pi^s}{\Gamma(s)\sqrt{a}} \sum_{n=1}^{\infty} n^{s-1/2} \cos(n\pi b/a) \sum_{d|n} d^{1-2s} \left(\Delta + \frac{4aq}{d^2}\right)^{-s/2+1/4} K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta + \frac{4aq}{d^2}}\right),$$

$\sum_{d|n} d^s$ denoting the sum over the divisors of n and where the function $\zeta_{EH}(s, p)$ (the one dimensional Epstein-Hurwitz or inhomogeneous Epstein series) is given by

$$\begin{aligned} \zeta_{EH}(s; p) &= \sum_{n=1}^{\infty} \left(n^2 + p\right)^{-s} = \frac{1}{2}\sum_{n\in\mathbf{Z}}{}' \left(n^2 + p\right)^{-s} \\ &= -\frac{p^{-s}}{2} + \frac{\sqrt{\pi}\,\Gamma(s-1/2)}{2\,\Gamma(s)} p^{-s+1/2} + \frac{2\pi^s p^{-s/2+1/4}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n\sqrt{p}). \end{aligned} \tag{3.9}$$

Eq. (3.8) provides the analytical continuation of the inhomogenous Epstein series, in the variable s, as a meromorphic function in the complex plane. Its pole structure is explicitly given in terms of the well-known pole structure of $\zeta_{EH}(s, p)$.

Eq. (3.8) has been found by one of us and called the *extended Chowla-Selberg* formula, since it contains the Chowla-Selberg formula as the particular case $q = 0$, i.e.,

$$E(s; a, b, c; 0) = 2\zeta(2s)\, a^{-s} + \frac{2^{2s}\sqrt{\pi}\, a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\, \Gamma(s-1/2)\zeta(2s-1) + \frac{2^{s+5/2}\pi^s}{\Gamma(s)\Delta^{s/2-1/4}\sqrt{a}}$$
$$\times \sum_{n=1}^{\infty} n^{s-1/2}\, \sigma_{1-2s}(n)\, \cos(n\pi b/a)\, K_{s-1/2}\left(\frac{\pi n \sqrt{\Delta}}{a}\right). \qquad (3.10)$$

where

$$\sigma_s(n) \equiv \sum_{d|n} d^s. \qquad (3.11)$$

Formula (3.8) has been obtained for the first time in [17], and a misprint has been corrected in [18]. The good convergence properties of expression (3.10), that were so much prised by Chowla and Selberg, are shared by its non-trivial extension (3.8). This renders the use of the formula quite simple. In fact, the two first terms are just nice —under the form (3.9)— while the last one (impressive in appearence) is even more quickly convergent than in the case of Eq. (3.10), and thus absolutely harmless in practice. Only a few first terms of the three series of Bessel functions in (3.8), (3.9) need to be calculated, even if one demands good accuracy. We should also notice that the only pole of (3.10) at $s = 1$ appears through $\zeta(2s-1)$ in the second term, while for $s = 1/2$, the apparent singularities of the first and second terms cancel each other and no pole is formed. Analogously, the pole at $s = 1/2$ in (3.8) comes only from the first term. Eq. (3.8) also has these good properties, *for any non-negative value of* q. In fact, for large q the convergence properties of the series of Bessel functions are clearly enhanced, while for q small we get back to the case of Chowla and Selberg. Notice, however, that this is not obtained through the high-q expansion (e.g., just putting $q = 0$ in (3.8)), but using a low-q, binomial expansion of the kind

$$\sum_{n=0}^{\infty} \left[a(n+c)^2 + q\right]^{-s} = a^{-s} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m+s)}{\Gamma(s)\, m!} \left(\frac{q}{a}\right)^m \zeta_H(2s+2m, c), \qquad (3.12)$$

which is convergent for $q/a \leq 1$. For $q \to 0$ it reduces to $a^{-s}\zeta_H(2s, c)$. Actually, formula (3.8) is still valid in a domain of negative q's, namely for $q > -\min(a, c, a-b+c)$.

Turning now to the particular application of the formula in our specific situation, we see that the zeta function corresponding to the Laplace-Beltrami operator in the massive (resp. massless) case is simply given by:

$$\zeta_{\Delta+m^2}(s) = m^{-2s} + \left(\frac{4\pi^2}{\tau_2}\right)^{-s} E\left(s; |\tau|^2, -2\tau_1, 1; \frac{\tau_2 m^2}{4\pi^2}\right) \qquad (3.13)$$

and

$$\zeta_{\Delta}(s) = \left(\frac{4\pi^2}{\tau_2}\right)^{-s} E\left(s; |\tau|^2, -2\tau_1, 1; 0\right), \qquad (3.14)$$

respectively. The values at $s=-1/2$ are finite and define the corresponding Casimir energy. After performing the necessary calculations, and according to the prescription that has been derived in the first part of this paper, we obtain as result the quite simple expressions

$$\zeta_{\Delta+m^2}(-1/2) = -\frac{m^3}{6\pi} - \frac{2m}{\pi}\sum_{n=1}^{\infty} n^{-1}K_1(nmx_2) - \sqrt{2}\left(\frac{mx_2}{\pi}\right)^{3/2}\sum_{n=1}^{\infty} n^{-3/2}K_{3/2}(nm/x_2)$$
$$-8x_2\sum_{n=1}^{\infty} n^{-1}\cos(2\pi n x_1^2)\sum_{d|n} d^2\sqrt{1+\frac{m^2}{(2\pi x_2 d)^2}}\; K_1\left(2\pi n x_2^2\sqrt{1+\frac{m^2}{(2\pi x_2 d)^2}}\right), \qquad (3.15)$$

and

$$\zeta_\Delta(-1/2) = -\frac{\pi}{3x_2} + 4\pi\zeta'(-2)x_2^3 - 8x_2\sum_{n=1}^{\infty} n^{-1}\cos(2\pi n x_1^2)\sigma_2(n)\, K_1\left(2\pi n x_2^2\right), \qquad (3.16)$$

in terms of the variables

$$x_1 = \sqrt{\frac{\tau_1}{\tau_1^2+\tau_2^2}}, \qquad x_2 = \sqrt{\frac{\tau_2}{\tau_1^2+\tau_2^2}}. \qquad (3.17)$$

The extrema of the corresponding Casimir energy for the case $m^2 = 100$, in terms of the original Teichmüller coefficients τ_1 and τ_2, are to be read from Figs. 1-3. In the three-dimensional plot over the plane τ_1, τ_2 (Fig. 1), the maximal Casimir energy is seen to be localized on the section $\tau_2 = 1$. In order to show this fact more clearly, in Fig. 2 we have represented the section $\tau_1 = 0$, but the situation is common to any section $\tau_1 =$ const. On the section $\tau_2 = 1$ a periodic structure appears asociated with the value of τ_1 along this section (Fig. 3). This behavior is easy to recognize from the form of the function $\zeta_\Delta(-1/2)$, (3.16), and is common to *any* section $\tau_2 =$ const. All the figures depicted here have been obtained taking the first 20 terms from the sum over n in (3.16).

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Figure captions

Fig. 1

Three-dimensional plot over the plane τ_1, τ_2, of the Casimir energy for the case $m = 10$. One clearly observes that the extremal Casimir energy is localized on the section $\tau_2 = 1$. All the figures have been obtained taking the first 20 terms from the sum over n in (3.16).

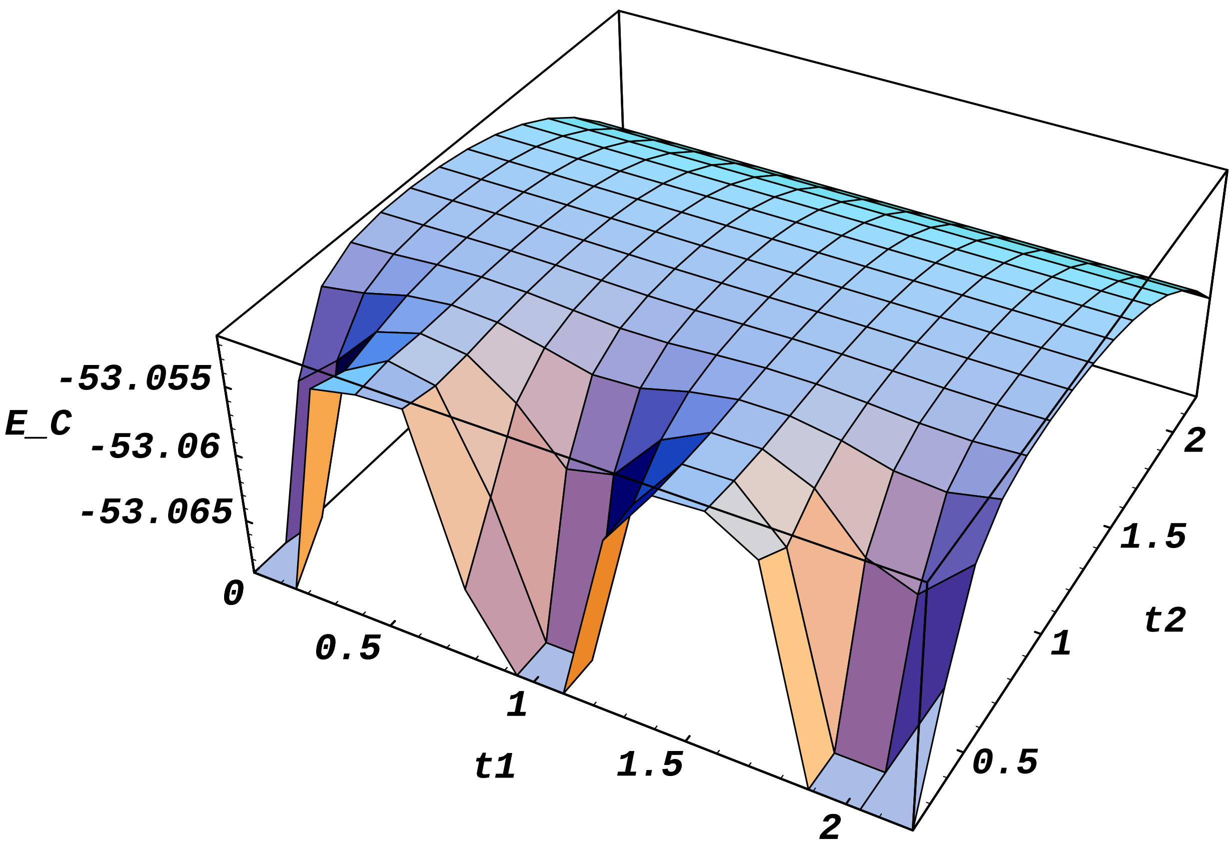
Fig. 2

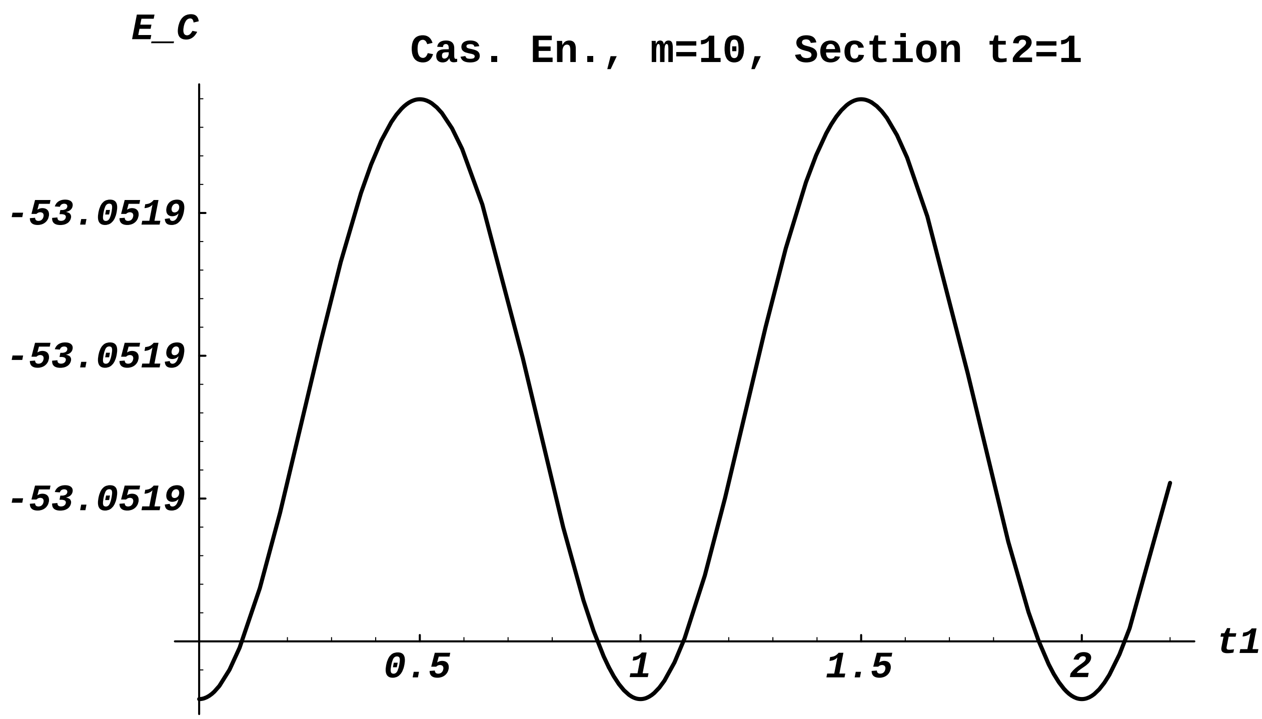
Plot of the section $\tau_1 = 0$, where the fact that the extremal Casimir energy is obtained for $\tau_2 = 1$ is seen undoubtedly. This behavior is common to any section $\tau_1 =$ const.

Fig. 3

Plot of the section $\tau_2 = 1$. Here a periodic structure appears asociated with the value of τ_1 along this section. This behavior is common to any section $\tau_2 =$ const.

Casimir Energy, m=10





E_C
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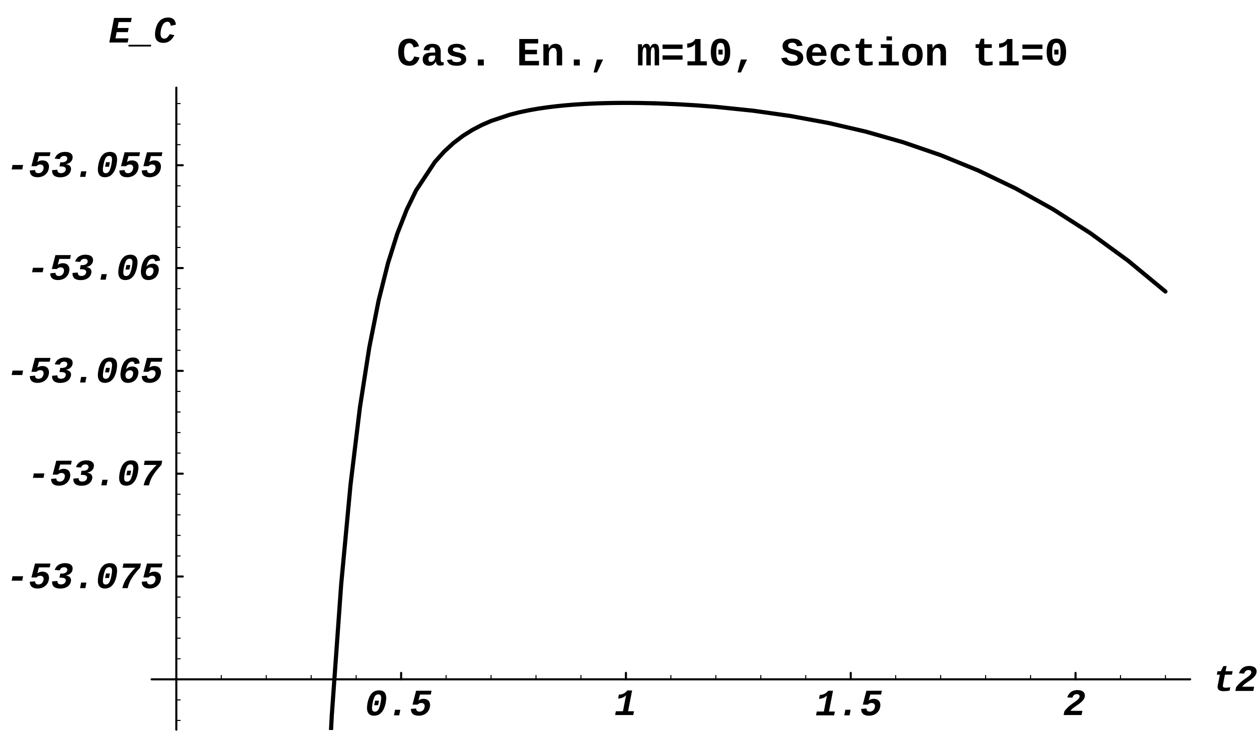
-53.0519
0.5
1
1.5
2
t1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0.5
1
1.5
2
t2